§2 Ansatz : Prequantisation

What is quantization?

Nothing more than a huge set of methods, principles and procedures to "construct" quantum systems by using classical systems. Common feature: No rules, no rigor, great freedom. Main objective: Arrive at a reasonable quantum system.

In particular: Quantization is not physics. There are no physical ideas or principles which support quantization on a rigorous level. The process

$$\text{classical system} \longmapsto \text{quantum system}$$

is speculation (although it generates remarkable examples). Only the reverse process

$$\text{quantum system} \longmapsto \text{classical system}$$

by taking classical or semi-classical limits can be justified by physical considerations.

Nevertheless, many important quantum systems (if not all) have been obtained by quantization.

Let's concentrate on quantum mechanics. The main quantum mechanical systems have been obtained by "canonical quantization".

What is Canonical Quantisation?

To quantize a classical system given by a Hamiltonian system (X, ω, H) requires according to Dirac to find a complex Hilbert space $\mathcal{H}$ and a map

$$q : \mathfrak{a} \longrightarrow \text{"End"}(\mathcal{H})$$

such that:

(D1) $q(1) = \lambda \, \mathrm{id}_{\mathcal{H}} = \lambda \cdot 1$ for a suitable constant $\lambda \neq 0$.

(D2) $q(\{f, g\}) = c[q(f), q(g)]$ for all $f, g \in \mathfrak{a}$.

In addition, all $q(f)$ shall be self-adjoint (possibly unbounded) linear operators on $\mathcal{H}$, and (D1) supposes $1 \in \mathfrak{a}$ and condition (D2) supposes that $\mathfrak{a} \subset \mathcal{E}(M)$ is a Lie subalgebra of the Poisson algebra $\mathcal{E}(M)$. The constant c is $\frac{i}{\hbar}$ or $-i$ or similar, depending on the conventions. From the mathematical point of view the value of c is irrelevant.

What counts is that all $q(f), f \in \mathfrak{a}$, can be recovered from a common domain $D \subset \mathcal{H}$ of definition, $D \subset \bigcap \{ D(q(f)) : f \in \mathfrak{a} \}$, which is a dense subspace of $\mathcal{H}$ so that

$$[q(f)|_D, q(g)|_D] \in \operatorname{End} D.$$

makes sense and coincides with

$$\hat{=} \; q(\{f,g\})|_D \, .$$

This is the meaning of "End" $\mathcal{H}$!

(D1) and (D2) are called Dirac conditions. In many contexts a quantization satisfying the above conditions is called canonical quantization. But there is no prescription how to obtain the Hilbert space $\mathcal{H}$ or q.

Geometric quantization is a canonical quantization in the sense above and gives a well-defined procedure how to find $\mathcal{H}$ and q. The first step in this procedure is PREQUANTIZATION and the purpose of this section is to motivate this concept.

Discussion: Essentially different (even controversal) usage of the term "canonical":
In physics: a special choice, in a standard or common way.
In maths: a natural way, not dependent on any choice. Universal, functorial.

We now come to the subject announced in the title of this section:

Ansatz: Prequantization

Let (M, ω) be a symplectic manifold.
We have a natural representation Φ of the Poisson algebra $(\mathcal{E}(M), \{\ \})$ by forming the hamiltonian vector field X_F of a function $F \in \mathcal{E}(M)$:

$$\Phi : \mathcal{E}(M) \longrightarrow \mathcal{V}(M) \quad \left(\cong \mathrm{Der}(\mathcal{E}(M)) \subset \mathrm{End}(\mathcal{E}(M)) \right)$$
$$F \longmapsto -X_F = \Phi(F) : \mathcal{E}(M) \to \mathcal{E}(M)$$

Recall that Φ is $\mathbb{R}$-linear and that for $F, G \in \mathcal{E}(M)$

$$\Phi(\{F, G\}) = -X_{\{F,G\}} = [X_F, X_G] = [\Phi(F), \Phi(G)]$$

(see (1.10)).

We have to complexify the whole machinery replacing every $\mathbb{R}$-vector space W which occured so far by $W \otimes_{\mathbb{R}} \mathbb{C} = W \oplus iW$. For example: $T_aM \otimes \mathbb{C}$ instead of T_aM, $\mathcal{E}(M) \otimes \mathbb{C} = \mathcal{E}(M, \mathbb{C})$, $\Omega^k(M) \otimes \mathbb{C}$ instead of $\Omega^k(M)$, $\mathfrak{or} \otimes \mathbb{C}$ instead of $\mathfrak{or}$ or we take a complex subalgebra $\mathfrak{or}$ of $\mathcal{E}(M, \mathbb{C})$, ...

Afterwards we omit $\mathbb{C}$ and in the following T_aM, $\Omega^k(M)$, $\mathcal{E}(M)$, $\mathfrak{or}$ ... shall denote the complexified versions.
Of course, $q : \mathfrak{or} \to$ "End"$(\mathcal{X})$ shall be complex linear with (D1) & (D2).

Let us now describe the "ansatz": As a first try let us consider $\tilde{q}(f) := -cX_f$, $f \in \mathcal{E}(M)$, although $\mathcal{E}(M)$ is not a Hilbert space:

<u>(2.1) PROPOSITION:</u> $\tilde{q}(f) : \mathcal{E}(M) \to \mathcal{E}(M)$ is $\mathbb{C}$-linear and satisfies

$$(D2) \qquad [\tilde{q}(F), \tilde{q}(G)] = c\,\tilde{q}(\{F,G\}), \quad F,G \in \mathcal{E}(M).$$

□ <u>Proof:</u> $[\tilde{q}(F), \tilde{q}(G)] = (-c)^2 [X_F, X_G] = c^2(-X_{\{F,G\}}) = c\,\tilde{q}(\{F,G\})$. □

But for $1 \in \mathcal{E}(M)$: $\tilde{q}(1) = 0$, hence (D1) is not satisfied.

In order to satisfy (D1), we can replace $\tilde{q}$ by

$$f \mapsto f + \tilde{q}(f), \quad f \in \mathcal{E}(M).$$

Now, (D1) is satisfied but (D2) is not. We make a further correction and arrive at

$$q(f) := f - cX_f + \alpha(X_f), \quad f \in \mathcal{E}(M),$$

with a 1-form $\alpha \in \Omega^1(M)$

<u>(2.2) PROPOSITION:</u> $q : \mathcal{E}(M) \to \mathcal{E}(M)$ is $\mathbb{C}$-linear and satisfies (D1). Moreover,

$$q \text{ fulfills (D2)} \iff d\alpha = \omega$$

□ <u>Proof:</u> $q(1) = 1$ since $X_1 = 0$; and q is $\mathbb{C}$-linear. For $F \in \mathcal{E}(M)$ let $\mu(F) = F + \alpha(X_F) \in \mathcal{E}(M)$ be the multiplication operator $\mu(F) : \mathcal{E}(M) \to \mathcal{E}(M)$, $H \longmapsto \mu(F)H$

Now

$$[q(F), q(G)] = [\tilde{q}(F) + \mu(F), \tilde{q}(G) + \mu(G)] =$$

$$= [\tilde{q}(F), \tilde{q}(G)] + [\tilde{q}(F), \mu(G)] + [\mu(F), \tilde{q}(G)] + [\mu(F), \mu(G)]$$

Hence,

$$[q(F), q(G)] = c\tilde{q}(\{F, G\}) + [\tilde{q}(F), \mu(G)] + [\mu(F), \tilde{q}(G)]$$

according to 2.1 and because of $[\mu(F), \mu(G)] = 0$.

For every $H \in \mathcal{E}(M)$

$$[\tilde{q}(F), \mu](H) = -c L_{X_F}(\mu \cdot H) + c\mu L_{X_F} H = -c(L_{X_F}\mu)H,$$

and we obtain

$$[\tilde{q}(F), \mu(G)] = -c L_{X_F} \mu(G) = c\{F, \mu(G)\}$$

In the same way

$$[\mu(F), \tilde{q}(G)] = c L_{X_G} \mu(F) = c\{\mu(F), G\}.$$

Altogether,

$$\begin{aligned}[q(F), q(G)] &= c\tilde{q}(\{F, G\}) + 2c\{F, G\} + c(\{F, \alpha(X_G)\} + \{\alpha(X_F), G\}) \\ &= c\,(\tilde{q}(\{F, G\}) + \{F, G\} + \alpha(X_{\{F, G\}})) + \\ &\quad + c\,(\{F, G\} + \{F, \alpha(X_G)\} + \{\alpha(X_F), G\} - \alpha(X_{\{F, G\}}))\end{aligned}$$

and we obtain

$$(*) \quad [q(F), q(G)] = c\, q(\{F, G\})$$

if and only if the term in the second bracket above vanishes, i.e. iff

$$\{F,G\} = \{\alpha(X_G),F\} - \{\alpha(X_F),G\} + \alpha(X_{\{F,G\}})$$

Now, from the formula for $d\alpha$:

$$d\alpha(X_F,X_G) = L_{X_F}\alpha(X_G) - L_{X_G}\alpha(X_F) - \alpha([X_F,X_G])$$
$$= \{\alpha(X_G),F\} - \{\alpha(X_F),G\} + \alpha(X_{\{F,G\}})$$

and the final result reads as claimed:

$$* \text{ holds} \iff d\alpha(X_F,X_G) = \{F,G\} = \omega(X_F,X_G) \;\forall F,G \in \mathcal{E}(M)$$
$$\iff d\alpha = \omega,$$

since locally the hamiltonian vector field generate the $\mathcal{E}(U)$-module of vector fields. $\square$

Because of this result, we assume for a moment that ω has a potential α : $d\alpha = \omega$. To proceed further, we observe that the $2n$-form

$$\omega^n = \omega \wedge \omega \wedge \ldots \wedge \omega$$

is a volume form. Let $\mathcal{H}$ be the complex Hilbert space which we obtain by completing the prehilbert space

$$\{f \in \mathcal{E}(M,\mathbb{C}) \mid \int_M |f|^2 d\omega^n < \infty\}$$

Then the $q(F)$, $F \in \mathcal{E}(M)$, which are defined on $\mathcal{H}$ (or suitable subspaces) satisfy D1 and D2. But there are 2 generic defects of the approach:

1. It does not work without symplectic potential α. For example not for $\mathbb{R}^n \times S^2$ ("spin") or $S^2 \times S^2$ (hydrogen atom). Note that for compact symplectic manifolds (M,ω) there

never exists a potential $\alpha \in \Omega^1(M)$ (i.e. $d\alpha = \omega$).
To manage this problem, one has to generalize the ansatz by replacing $\mathcal{E}(M)$ by the sections of a complex line bundle over M. This will be explained in the next section.

2. $\mathcal{H}$ is generated by functions in $2n$ variables. This should be reduced to n variables. This can be done by introducing polarisations. This will come in a later section.

Apart from 1. and 2. one has to find the correct Hilbert space in a unique manner and then one has to check for which F the $q(F)$ are selfadjoint, among other considerations.

Let us study the case $M = T^*Q$, $Q \subset \mathbb{R}^n$ open (or an n-dim. manifold). Canonical coordinates: q^j & p_j.

$\omega(X_{q^j}, Y) = dq^j(Y)$ & $\omega = dq^j \wedge dp_j$ yields

$$dq^j(Y) = dq^k(X_{q^j})dp_k(Y) - dq^k(Y)dp_k(X_{q^j})$$

$$X_{q^j} = -\frac{\partial}{\partial p_j} \quad \text{and correspondingly} \quad X_{p_j} = \frac{\partial}{\partial q^j}$$

$$q(q^j) = +c\frac{\partial}{\partial p_j} + q^j =: Q^j$$

$$q(p_j) = -c\frac{\partial}{\partial q^j} + p_j - p_j := P_j$$

$$[Q^j, P_k] = c\,q(\{q^j, p_k\}) = c\,\delta^j_k \quad (X_{\{q^j,p_k\}} = 0)$$

By replacing the Hilbert space by its subspace of all functions f of the form $f = g \circ \tau$, $g : Q \to \mathbb{C}$, for suitable g we arrive at a function space with the correct dependencies and moreover:

$$Q^j = q^j \quad \text{and} \quad P_j = -c\frac{\partial}{\partial q^j}.$$